

OKOUNKOV BODIES ASSOCIATED TO PSEUDOEFFECTIVE DIVISORS II

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ABSTRACT. We first prove some basic properties of Okounkov bodies, and give a characterization of Nakayama and positive volume subvarieties of a pseudoeffective divisor in terms of Okounkov bodies. Next, we show that each valutive and limiting Okounkov bodies of a pseudoeffective divisor which admits the birational good Zariski decomposition is a rational polytope with respect to some admissible flag. This is an extension of the result of Anderson-Küronya-Lozovanu about the rational polyhedrality of Okounkov bodies of big divisors with finitely generated section rings.

1. INTRODUCTION

This paper is a continuation of our investigation on Okounkov bodies associated to pseudoeffective divisors ([CHPW1], [CHPW2], [CPW]). Let X be a smooth projective variety of dimension n , and D be a divisor on X . Fix an admissible flag Y_\bullet on X , that is, a sequence of irreducible subvarieties

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{x\}$$

where each Y_i is of codimension i in X and is smooth at x . The Okounkov body $\Delta_{Y_\bullet}(D)$ of a big divisor D with respect to Y_\bullet is a convex body in the Euclidean space \mathbb{R}^n which carries rich information of D . Okounkov first defined the Okounkov body associated to an ample divisor in [O1], [O2]. After this pioneering work, Lazarsfeld-Mustařă [LM] and Kaveh-Khovanskii [KK] independently generalized Okounkov's work to big divisors (see [B2] for a survey). We then further extended the study of Okounkov bodies to pseudoeffective divisors in [CHPW1]. More precisely, we have introduced and studied two convex bodies, called the *valuative Okounkov body* $\Delta_{Y_\bullet}^{\text{val}}(D)$ and the *limiting Okounkov body* $\Delta_{Y_\bullet}^{\text{lim}}(D)$ associated to a pseudoeffective divisor D . See Sections 3 and 4 for definitions and basics on Okounkov bodies.

In this paper, we first prove supplementary results to [CHPW1]. Main theorems of [CHPW1] and the subsequent results in this paper depend on the following property of the Okounkov body. This theorem is a generalization of [LM, Theorem 4.26] and [J, Theorem 3.4].

Theorem A (=Theorem 3.6). *Let X be a smooth projective variety of dimension n , and D be a big divisor on X . Fix an admissible flag Y_\bullet such that $Y_{n-k} \not\subseteq \mathbf{B}_+(D)$. Then we have*

$$\Delta_{Y_{n-k,\bullet}}(D) = \Delta_{Y_\bullet}(D) \cap (\{0\}^{n-k} \times \mathbb{R}_{\geq 0}^k).$$

In [CHPW1], we proved that the Okounkov bodies $\Delta_{Y_\bullet}^{\text{val}}(D)$ and $\Delta_{Y_\bullet}^{\text{lim}}(D)$ encode nice properties of the divisor D if the given admissible flag Y_\bullet contains a Nakayama subvariety of D or a positive volume subvariety of D (see Theorem 4.6). We show the following characterization of those special subvarieties in terms of Okounkov bodies.

Theorem B (=Theorem 4.8). *Let X be a smooth projective variety of dimension n , and D be an \mathbb{R} -divisor on X . Fix an admissible flag Y_\bullet such that Y_n is a general point in X . Then we have the following:*

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- (1) If D is effective, then Y_\bullet contains a Nakayama subvariety of D if and only if $\Delta_{Y_\bullet}^{\text{val}}(D) \subseteq \{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$.
- (2) If D is pseudoeffective, then Y_\bullet contains a positive volume subvariety of D if and only if $\Delta_{Y_\bullet}^{\text{lim}}(D) \subseteq \{0\}^{n-\kappa_\nu(D)} \times \mathbb{R}^{\kappa_\nu(D)}$ and $\dim \Delta_{Y_\bullet}^{\text{lim}}(D) = \kappa_\nu(D)$.

One of the most important properties one can probably expect a convex set in \mathbb{R}^n to satisfy is rational polyhedrality. However, the geometric structure of Okounkov body is rather wild. It can be non-polyhedral even if the variety X is a Mori dream space and a divisor D is ample (see [LM, Subsection 6.3], [KLM, Section 3]). However, Anderson-Küronya-Lozovanu proved that if a big divisor D has a finitely generated section ring $R(X, D) := \bigoplus_{m \geq 0} H^0(X, mD)$, then there exists an admissible flag Y_\bullet such that the Okounkov body $\Delta_{Y_\bullet}(D)$ is a rational polytope ([AKL, Theorem 1]). We also refer to [CPW, Theorems 1.1 and 4.17] and [S, Corollary 4.5] for more related results.

Our next aim is to generalize [AKL, Theorem 1] to the valuative and limiting Okounkov bodies. We recall that when a divisor D is big, it has a finitely generated section ring if and only if it admits the birational good Zariski decomposition (see [N, III.1.17.Remark]). However, for a pseudoeffective divisor D , such equivalence no longer holds in general; D admits the birational good Zariski decomposition if and only if D has a finitely generated section ring and is abundant (see Proposition 2.7). For the rational polyhedrality of the Okounkov bodies of pseudoeffective divisors, we assume the existence of good Zariski decomposition on some birational model instead of the finite generation condition. See Subsection 2.3 for our definition of (good) Zariski decomposition.

Theorem C (=Corollary 5.4 and Theorem 5.6). *Let X be a smooth projective variety, and D be a pseudoeffective \mathbb{Q} -divisor on X which admits the good birational Zariski decomposition. Then each Okounkov bodies $\Delta_{Y_\bullet}^{\text{val}}(D)$ and $\Delta_{Y_\bullet}^{\text{lim}}(D)$ is rational polyhedral with respect to some admissible flag Y_\bullet .*

We expect that the rational polyhedrality of Okounkov body holds in more general situations. There are examples of divisors which do not admit birational good Zariski decompositions, but whose associated Okounkov bodies are rational polyhedral (see Remark 5.7).

To prove Theorem C for the case of valuative Okounkov bodies, we use the same idea as [AKL, Proposition 4]. Using only the finite generation of section ring, we show the rational polyhedrality of the valuative Okounkov body with respect to an admissible flag taken by the intersections of general members of the linear series (see Theorem 5.3). For the case of limiting Okounkov bodies, under the given assumption, we prove the statement by reducing to the rationality problem of the limiting Okounkov body on some high model $f: Y \rightarrow X$ where the good Zariski decomposition of f^*D exists (see Theorem 5.6).

The organization of the paper is as follows. In Section 2, we collect basic facts on various notions that are used in the proofs. Next, in Section 3, we recall basic properties of Okounkov bodies, and prove Theorem A. Then we study some properties of Nakayama subvarieties and positive volume subvarieties to show Theorem B in Section 4. Section 5 is devoted to showing Theorem C.

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2. PRELIMINARIES

In this section, we collect relevant facts which will be used later. Throughout the paper, X is a smooth projective variety of dimension n , and we always work over an algebraically closed field of characteristic zero.

2.1. Asymptotic invariants. We review basic asymptotic invariants of divisors, namely, the asymptotic base loci and volume functions. The *stable base locus* of an \mathbb{R} -divisor D is defined as $\text{SB}(D) := \bigcap_{D \sim_{\mathbb{R}} D' \geq 0} \text{Supp}(D')$. The *augmented base locus* of an \mathbb{R} -divisor D is defined as $\mathbf{B}_+(D) := \bigcap_A \text{SB}(D - A)$ where the intersection is taken over all ample divisors A . The *restricted base locus* of an \mathbb{R} -divisor D is defined as $\mathbf{B}_-(D) := \bigcup_A \text{SB}(D + A)$ where the union is taken over all ample divisors A . Note that $\mathbf{B}_+(D)$ and $\mathbf{B}_-(D)$ depend only on the numerical class of D . For details, we refer to [ELMNP1] and [Le2].

Now, let V be an irreducible subvariety of X of dimension v . The *restricted volume* of a \mathbb{Z} -divisor D along V is defined as $\text{vol}_{X|V}(D) := \limsup_{m \rightarrow \infty} \frac{h^0(X|V, mD)}{m^{v/v!}}$ where $h^0(X|V, mD)$ is the dimension of the image of the natural restriction map $\varphi: H^0(S, \mathcal{O}_X(D)) \rightarrow H^0(V, \mathcal{O}_V(D))$. The restricted volume $\text{vol}_{X|V}(D)$ depends only on the numerical class of D , and one can uniquely extend it to a continuous function

$$\text{vol}_{X|V}: \text{Big}^V(X) \rightarrow \mathbb{R}$$

where $\text{Big}^V(X)$ is the set of all \mathbb{R} -divisor classes ξ such that V is not properly contained in any irreducible component of $\mathbf{B}_+(\xi)$. When $V = X$, we simply let $\text{vol}_X(D) := \text{vol}_{X|X}(D)$, and we call it the *volume* of an \mathbb{R} -divisor D . For more details on volumes and restricted volumes, see [La] and [ELMNP2]. Now assume that $V \not\subseteq \mathbf{B}_-(D)$ for an \mathbb{R} -divisor D . The *augmented restricted volume* of D along V is defined as $\text{vol}_{X|V}^+(D) := \lim_{\varepsilon \rightarrow 0+} \text{vol}_{X|V}(D + \varepsilon A)$ where A is an ample divisor on X . The definition is independent of the choice of A . Note that $\text{vol}_{X|V}^+(D) = \text{vol}_{X|V}(D)$ for $D \in \text{Big}^V(X)$. This also extends uniquely to a continuous function

$$\text{vol}_{X|V}^+: \overline{\text{Eff}}^V(X) \rightarrow \mathbb{R}$$

where $\overline{\text{Eff}}^V(X) := \text{Big}^V(X) \cup \{\xi \in \overline{\text{Eff}}(X) \setminus \text{Big}(X) \mid V \not\subseteq \mathbf{B}_-(\xi)\}$. For $D \in \overline{\text{Eff}}^V(X)$, we have $\text{vol}_{X|V}(D) \leq \text{vol}_{X|V}^+(D) \leq \text{vol}_V(D|_V)$, and both inequalities can be strict in general. See [CHPW1] for more details on augmented restricted volumes.

2.2. Iitaka dimension. Let D be an \mathbb{R} -divisor on X . Let $\mathbb{N}(D) = \{m \in \mathbb{Z}_{>0} \mid |[mD]| \neq \emptyset\}$. For $m \in \mathbb{N}(D)$, we consider the rational map $\phi_{mD}: X \dashrightarrow \mathbb{P}^{\dim |[mD]|}$ defined by the linear system $|[mD]|$. The *Iitaka dimension* of D is defined as

$$\kappa(D) := \begin{cases} \max\{\dim \text{Im}(\phi_{mD}) \mid m \in \mathbb{N}(D)\} & \text{if } \mathbb{N}(D) \neq \emptyset \\ -\infty & \text{if } \mathbb{N}(D) = \emptyset. \end{cases}$$

We remark that the Iitaka dimension $\kappa(D)$ is not really an invariant of the \mathbb{R} -linear equivalence class of D . Nonetheless, it satisfies the property that $\kappa(D) = \kappa(D')$ for effective divisors D, D' such that $D \sim_{\mathbb{R}} D'$.

For another important invariant, we fix a sufficiently ample \mathbb{Z} -divisor A on X . The *numerical Iitaka dimension* of D is defined as

$$\kappa_{\nu}(D) := \max \left\{ k \in \mathbb{Z}_{\geq 0} \mid \limsup_{m \rightarrow \infty} \frac{h^0(X, [mD] + A)}{m^k} > 0 \right\}$$

if $h^0(X, [mD] + A) \neq \emptyset$ for infinitely many $m > 0$, and we let $\kappa_{\nu}(D) := -\infty$ otherwise. The numerical Iitaka dimension $\kappa_{\nu}(D)$ depends only on the numerical class $[D] \in \mathbb{N}^1(X)_{\mathbb{R}}$.

Definition 2.1. An \mathbb{R} -divisor D is said to be *abundant* if $\kappa(D) = \kappa_{\nu}(D)$.

By definition, $\kappa(D) \leq \kappa_{\nu}(D)$ holds and the inequality can be strict in general. However, $\kappa_{\nu}(D) = \dim X$ if and only if $\kappa(D) = \dim X$. We refer to [E], [Le1], [N] for more detailed properties of κ and κ_{ν} .

Recall that the *section ring* of an \mathbb{R} -divisor D is defined as $R(X, D) := \bigoplus_{m \geq 0} H^0(X, [mD])$.

Proposition 2.2 ([MR, Corollary 1]). *A \mathbb{Q} -divisor D on X is semiample if and only if it is nef, abundant, and its section ring is finitely generated.*

2.3. Zariski decomposition. We now briefly recall several notions related to Zariski decompositions in higher dimension. For more details, we refer to [B1], [N], [P].

To define the divisorial Zariski decomposition, we first consider a divisorial valuation σ on X with the center $V := \text{Cent}_X \sigma$ on X . If D is a big \mathbb{R} -divisor on X , we define the *asymptotic valuation* of σ at D as $\text{ord}_V(\|D\|) := \inf\{\sigma(D') \mid D \equiv D' \geq 0\}$. If D is only a pseudoeffective \mathbb{R} -divisor on X , we define $\text{ord}_V(\|D\|) := \lim_{\varepsilon \rightarrow 0+} \text{ord}_V(\|D + \varepsilon A\|)$ for some ample divisor A on X . This definition is independent of the choice of A . The *divisorial Zariski decomposition* of a pseudoeffective \mathbb{R} -divisor D is the decomposition

$$D = P_\sigma + N_\sigma$$

into the *negative part* $N_\sigma := \sum_{\text{codim } E=1} \text{ord}_E(\|D\|)E$ where the summation is over the codimension one irreducible subvarieties E of X such that $\text{ord}_E(\|D\|) > 0$ and the *positive part* $P_\sigma := D - N_\sigma$.

Let D be an \mathbb{R} -divisor on X which is effective up to $\sim_{\mathbb{R}}$. The *s-decomposition* of D is the decomposition

$$D = P_s + N_s$$

into the *negative part* $N_s := \inf\{L \mid L \sim_{\mathbb{R}} D, L \geq 0\}$ and the *positive part* $P_s := D - N_s$. The positive part P_s is also characterized as the smallest divisor such that $P_s \leq D$ and $R(X, P_s) \simeq R(X, D)$ ([P, Proposition 4.8]). Note that $P_s \leq P_\sigma$ and P_s, P_σ do not coincide in general.

Lemma 2.3. *Let D be an abundant \mathbb{R} -divisor on X with the divisorial Zariski decomposition $D = P_\sigma + N_\sigma$ and the s-decomposition $D = P_s + N_s$. Then $P_\sigma = P_s$.*

Proof. Let σ be a divisorial valuation on X with $V = \text{Cent}_X \sigma$. [Le2, Proposition 6.4] implies that $\inf_{m \in \mathbb{Z}_{>0}, D' \in \lfloor mD \rfloor} \frac{1}{m} \sigma(D) = \text{ord}_V(\|D\|)$ holds. Since $\inf_{m \in \mathbb{Z}_{>0}, D' \in \lfloor mD \rfloor} \frac{1}{m} \sigma(D) = \sigma(N_s)$, we see that $D = P_s + N_s$ is the divisorial Zariski decomposition. \square

The *Fujita-Zariski decomposition* of a pseudoeffective \mathbb{R} -divisor D is the decomposition

$$D = P_f + N_f$$

into the effective *negative part* N_f and the nef *positive part* P_f such that if $f: Y \rightarrow X$ is a birational morphism from a smooth projective variety and $f^*D = P' + N'$ with P' nef and $N' \geq 0$, then $P' \leq f^*P$. By definition, the divisorial Zariski decomposition and s-decomposition uniquely exist, and the Fujita-Zariski decomposition is also unique if it exists. Recall that the Fujita-Zariski decomposition does not exist in general even if we take the pullback on a sufficiently high model $f: \tilde{X} \rightarrow X$ (see [N, Chapter IV]).

It is unclear in general whether the Fujita-Zariski decomposition is the divisorial Zariski decomposition (cf. [N, III.1.17.Remark (2)]). However, this holds when the divisor is abundant and the positive part is semiample.

Proposition 2.4. *Let D be an abundant \mathbb{Q} -divisor on X having a decomposition $D = P + N$ into a nef divisor P and an effective divisor N . Then the following are equivalent:*

- (1) *It is the divisorial Zariski decomposition with $P = P_\sigma$ semiample.*
- (2) *It is the Fujita-Zariski decomposition with $P = P_f$ semiample.*
- (3) *It is the s-decomposition with $P = P_s$ semiample.*

Proof. (1) \Rightarrow (2): It is easy to check that the divisorial Zariski decomposition with the nef positive part is the Fujita-Zariski decomposition (see [N, III.1.17.Remark]).

(2) \Rightarrow (3): Let $D = P_s + N_s$ be the s-decomposition. Then $P_f \geq P_s$ by definition. Since P_f is semiample, we also have $P_f \leq P_s$. Therefore $P_f = P_s$.

(3) \Rightarrow (1): It follows from Lemma 2.3. \square

Definition 2.5. If one of the conditions in Proposition 2.4 holds for an abundant \mathbb{Q} -divisor D , then we say that D *admits the good Zariski decomposition*, and denote it by $D = P + N$. We say that D *admits the birational good Zariski decomposition* if there exists a birational morphism $f: \tilde{X} \rightarrow X$ from a smooth projective variety such that f^*D admits the good Zariski decomposition.

Proposition 2.6. *Let D be a pseudoeffective \mathbb{Q} -divisor with the good Zariski decomposition $D = P + N$. Then P, N are also \mathbb{Q} -divisors.*

Proof. Since P is semiample, there exists a morphism $f: X \rightarrow Y$ such that $P \sim_{\mathbb{R}} f^*A$ where A is an ample divisor on Y . The ample divisor A can be written as a finite sum of ample Cartier divisors on Y with positive real coefficients. Thus we can write $P \sim_{\mathbb{R}} \sum_{i=1}^k a_i P_i$ for some semiample Cartier divisors P_i and some positive real numbers a_i . Now we write $N = \sum_{j=1}^m b_j N_j$ for prime divisors N_1, \dots, N_m and positive real numbers b_j . Then N_1, \dots, N_m are linearly independent in $N^1(X)_{\mathbb{R}}$ by [N, III.1.10.Proposition]. Let V_P and V_N be the subspaces of $N^1(X)_{\mathbb{R}}$ spanned by $\{P_i\}_{i=1}^k$ and $\{N_j\}_{j=1}^m$, respectively. We now claim that $V_P \cap V_N = \{0\}$. Suppose that the claim does not hold. Then there exists a nonzero class $\eta \in V_P \cap V_N$ such that $\eta \equiv P' \equiv N'$ where $P' \in \bigoplus_{i=1}^k \mathbb{R} \cdot P_i$ and $N' \in \bigoplus_{j=1}^m \mathbb{R} \cdot N_j$. Note that there exists a positive number $\varepsilon > 0$ such that for any real number r satisfying $|r| < \varepsilon$, the divisor $P - rP'$ is nef and $N + rN'$ is effective. Thus [N, Proposition III.1.14 (2)] implies that in the following decompositions

$$\begin{aligned} D &= P + N \\ &\equiv (P - rP') + (N + rN'), \end{aligned}$$

we have $N \leq N + rN'$, hence $0 \leq rN'$ for any r such that $|r| < \varepsilon$. However, since N' is a nonzero divisor, this is a contradiction. The claim implies that if D is a \mathbb{Q} -divisor, then so is N in the decomposition $D = P + N$. Therefore P, N are both \mathbb{Q} -divisors. \square

Now, we characterize when a divisor admits the birational good Zariski decomposition.

Proposition 2.7. *Let D be a pseudoeffective \mathbb{Q} -divisor on X . Then D admits the birational good Zariski decomposition if and only if D is abundant and $R(X, D)$ is finitely generated.*

Proof. Suppose that there exists a birational morphism $f: \tilde{X} \rightarrow X$ from a smooth projective variety such that $f^*D = P + N$ is the good Zariski decomposition. By definition, D is abundant. Note that $R(X, D) \simeq R(\tilde{X}, f^*D) \simeq R(\tilde{X}, P)$. Since P is a semiample \mathbb{Q} -divisor by Proposition 2.6, it follows from Proposition 2.2 that $R(X, D)$ is finitely generated. Conversely, suppose that D is abundant and $R(X, D)$ is finitely generated. For a sufficiently large and divisible integer $m > 0$, we take a resolution $f: \tilde{X} \rightarrow X$ of the base locus of $|mD|$ and consider the decomposition $f^*(mD) = M + F$ into the base point free M and the fixed part F of $|f^*mD|$. By the finite generation of $R(X, D)$, we see that $f^*D = \frac{1}{m}M + \frac{1}{m}F$ is the s -decomposition with semiample positive part. By Proposition 2.4, f^*D admits the good Zariski decomposition. \square

3. OKOUNKOV BODIES

In this section, we recall the construction of Okounkov bodies associated to pseudoeffective divisors in [LM], [KK], and [CHPW1] and basic results. In the end, we prove Theorem A (=Theorem 3.6).

First, fix an admissible flag on X

$$Y_{\bullet}: X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_{n-1} \supseteq Y_n = \{x\}$$

where each Y_i is an irreducible subvariety of codimension i in X and is smooth at x . Let D be an \mathbb{R} -divisor on X with $|D|_{\mathbb{R}} := \{D' \mid D \sim_{\mathbb{R}} D' \geq 0\} \neq \emptyset$. We define a valuation-like function

$$\nu_{Y_{\bullet}}: |D|_{\mathbb{R}} \rightarrow \mathbb{R}_{\geq 0}^n$$

as follows. For $D' \in |D|_{\mathbb{R}}$, let

$$\nu_1 = \nu_1(D') := \text{ord}_{Y_1}(D').$$

Since $D' - \nu_1(D')Y_1$ is effective, we can define

$$\nu_2 = \nu_2(D') := \text{ord}_{Y_2}((D' - \nu_1 Y_1)|_{Y_1}).$$

If $\nu_i = \nu_i(D')$ is defined, then we define $\nu_{i+1} = \nu_{i+1}(D')$ inductively as

$$\nu_{i+1}(D') := \text{ord}_{Y_{i+1}}((\cdots ((D' - \nu_1 Y_1)|_{Y_1} - \nu_2 Y_2)|_{Y_2} - \cdots - \nu_i Y_i)|_{Y_i}).$$

The values $\nu_i(D')$ for $1 \leq i$ obtained as above define $\nu_{Y_{\bullet}}(D') = (\nu_1(D'), \nu_2(D'), \dots, \nu_n(D'))$.

Definition 3.1. The *Okounkov body* $\Delta_{Y_{\bullet}}(D)$ of a big \mathbb{R} -divisor D with respect to an admissible flag Y_{\bullet} is defined as the closure of the convex hull of $\nu_{Y_{\bullet}}(|D|_{\mathbb{R}})$ in $\mathbb{R}_{\geq 0}^n$.

More generally, a similar construction can be applied to a graded linear series W_{\bullet} on X to construct the Okounkov body $\Delta_{Y_{\bullet}}(W_{\bullet})$ of W_{\bullet} . For more details, we refer to [LM].

When D is not big, we have the following extension introduced in [CHPW1].

Definition 3.2 ([CHPW1, Definitions 1.1 and 1.2]). Let D be an \mathbb{R} -divisor on X .

- (1) When D is effective up to $\sim_{\mathbb{R}}$, i.e., $|D|_{\mathbb{R}} \neq \emptyset$, the *valuative Okounkov body* $\Delta_{Y_{\bullet}}^{\text{val}}(D)$ of D with respect to an admissible flag Y_{\bullet} is defined as the closure of the convex hull of $\nu_{Y_{\bullet}}(|D|_{\mathbb{R}})$ in $\mathbb{R}_{\geq 0}^n$. If $|D|_{\mathbb{R}} = \emptyset$, then we set $\Delta_{Y_{\bullet}}^{\text{val}}(D) := \emptyset$.
- (2) When D is pseudoeffective, the *limiting Okounkov body* $\Delta_{Y_{\bullet}}^{\text{lim}}(D)$ of D with respect to an admissible flag Y_{\bullet} is defined as

$$\Delta_{Y_{\bullet}}^{\text{lim}}(D) := \lim_{\varepsilon \rightarrow 0+} \Delta_{Y_{\bullet}}(D + \varepsilon A) = \bigcap_{\varepsilon > 0} \Delta_{Y_{\bullet}}(D + \varepsilon A),$$

where A is an ample divisor on X . (Note that $\Delta_{Y_{\bullet}}^{\text{lim}}(D)$ is independent of the choice of A .) If D is not pseudoeffective, we set $\Delta_{Y_{\bullet}}^{\text{lim}}(D) := \emptyset$.

Remark 3.3. Boucksom's numerical Okounkov body $\Delta_{Y_{\bullet}}^{\text{num}}(D)$ in [B2] is the same as our limiting Okounkov body $\Delta_{Y_{\bullet}}^{\text{lim}}(D)$.

Suppose that D is effective. By definition, $\Delta_{Y_{\bullet}}^{\text{val}}(D) \subseteq \Delta_{Y_{\bullet}}^{\text{lim}}(D)$, and the inclusion can be strict in general (see [CHPW1, Examples 4.2 and 4.3]). Moreover, by [B2, Proposition 3.3 and Lemma 4.8], we have

$$\dim \Delta_{Y_{\bullet}}^{\text{val}}(D) = \kappa(D) \leq \dim \Delta_{Y_{\bullet}}^{\text{lim}}(D) \leq \kappa_{\nu}(D).$$

The following lemmas will be useful for computing Okounkov bodies.

Lemma 3.4. Let D be an \mathbb{R} -divisor on X . Consider a birational morphism $f : \tilde{X} \rightarrow X$ with \tilde{X} smooth and an admissible flag

$$\tilde{Y}_{\bullet} : \tilde{X} = \tilde{Y}_0 \supseteq \tilde{Y}_1 \supseteq \cdots \supseteq \tilde{Y}_{n-1} \supseteq \tilde{Y}_n = \{x'\}.$$

on \tilde{X} . Suppose that Y_n is a general point in X and

$$Y_{\bullet} := f(\tilde{Y}_{\bullet}) : X = Y_0 \supseteq Y_1 = f(\tilde{Y}_1) \supseteq \cdots \supseteq Y_{n-1} = f(\tilde{Y}_{n-1}) \supseteq Y_n = f(\tilde{Y}_n) = \{f(x')\}.$$

is an admissible flag on X . Then we have $\Delta_{Y_{\bullet}}^{\text{val}}(f^*D) = \Delta_{Y_{\bullet}}^{\text{val}}(D)$ and $\Delta_{Y_{\bullet}}^{\text{lim}}(f^*D) = \Delta_{Y_{\bullet}}^{\text{lim}}(D)$.

Proof. The limiting Okounkov body case is shown in [CHPW2, Lemma 3.3]. The proof for the valuative Okounkov body case is almost identical and we leave the details to the readers as an exercise. \square

Lemma 3.5. Let D be an \mathbb{R} -divisor on X with the s -decomposition $D = P_s + N_s$ and the divisorial Zariski decomposition $D = P_{\sigma} + N_{\sigma}$. Fix an admissible flag Y_{\bullet} on X such that Y_n is a general point in X . Then we have $\Delta_{Y_{\bullet}}^{\text{val}}(D) = \Delta_{Y_{\bullet}}^{\text{val}}(P_s)$ and $\Delta_{Y_{\bullet}}^{\text{lim}}(D) = \Delta_{Y_{\bullet}}^{\text{lim}}(P_{\sigma})$, respectively.

Proof. The first assertion follows from the fact that $R(X, D) \simeq R(X, P_s)$ and the construction of the valuative Okounkov body. The second assertion is nothing but [CHPW2, Lemma 3.5]. \square

Finally, we give a proof of the main result of this section. The following key result is implicitly used in [CHPW1] (especially in the proof of [CHPW1, Theorem B]) and in this paper as well. We include the complete proof here.

Theorem 3.6. *Let X be a smooth projective variety of dimension n , and D be a big divisor on X . Fix an admissible flag Y_\bullet such that $Y_{n-k} \not\subseteq \mathbf{B}_+(D)$. Then we have*

$$\Delta_{Y_{n-k}\bullet}(D) = \Delta_{Y_\bullet}(D) \cap (\{0\}^{n-k} \times \mathbb{R}_{\geq 0}^k).$$

Proof. We may assume that each Y_i is a smooth variety. Let $\{A_i\}$ be a sequence of ample divisors on X such that each $D + A_i$ is a \mathbb{Q} -divisor and $\lim_{i \rightarrow \infty} A_i = 0$. Then we have

$$\Delta_{Y_\bullet}(D) = \bigcap_{i=1}^{\infty} \Delta_{Y_\bullet}(D + A_i) \text{ and } \Delta_{Y_{n-k}\bullet}(D) = \bigcap_{i=1}^{\infty} \Delta_{Y_{n-k}\bullet}(D + A_i).$$

Furthermore, $Y_{n-k} \not\subseteq \mathbf{B}_+(D + A_i)$ for all i . Note that it is enough to prove the statement for the \mathbb{Q} -divisors $D + A_i$ for all sufficiently large i . Thus we assume below that D is a \mathbb{Q} -divisor.

It is easy to check that $\Delta_{Y_{n-k}\bullet}(D) \subseteq \Delta_{Y_\bullet}(D)$. This implies that $\Delta_{Y_{n-k}\bullet}(D) \subseteq \Delta_{Y_\bullet}(D) \cap (\{0\}^{n-k} \times \mathbb{R}_{\geq 0}^k)$ by definition. Suppose that the inclusion is strict:

$$\Delta_{Y_{n-k}\bullet}(D) \subsetneq \Delta_{Y_\bullet}(D) \cap (\{0\}^{n-k} \times \mathbb{R}_{\geq 0}^k).$$

Then there exists a point $(0^{n-k}, x_1, \dots, x_k) \in \Delta_{Y_\bullet}(D) \cap (\{0\}^{n-k} \times \mathbb{R}_{\geq 0}^k)$, but $(0^{n-k}, x_1, \dots, x_k) \notin \Delta_{Y_{n-k}\bullet}(D)$.

Let A be an ample \mathbb{Q} -divisor on X . Note that $\Delta_{Y_{n-k}\bullet}(D) \subseteq \Delta_{Y_{n-k}\bullet}(D + \varepsilon A)$ for any $\varepsilon \geq 0$. Since $Y_{n-k} \not\subseteq \mathbf{B}_+(D + \varepsilon A)$, we have $\text{vol}_{\mathbb{R}^k} \Delta_{Y_{n-k}\bullet}(D + \varepsilon A) = \frac{1}{(n-k)!} \text{vol}_{X|Y_{n-k}}(D + \varepsilon A)$. Recall that by [ELMNP2, Theorem A], the function $\text{vol}_{X|Y_{n-k}} : \text{Big}^{Y_{n-k}}(X) \rightarrow \mathbb{R}$ is continuous, where $\text{Big}^{Y_{n-k}}(X)$ denotes the cone in $N^1(X)_{\mathbb{R}}$ consisting of the real divisor classes η such that Y_{n-k} is not properly contained in any of the irreducible components of $\mathbf{B}_+(\eta)$. Thus we can find a rational number $\varepsilon > 0$ such that $(x_1, \dots, x_k) \notin \Delta_{Y_{n-k}\bullet}(D + \varepsilon A)$ and

$$\text{vol}_{\mathbb{R}^k} \Delta_{Y_{n-k}\bullet}(D + \varepsilon A) < \text{vol}_{\mathbb{R}^k} \Delta$$

where $\Delta \subseteq \mathbb{R}^k$ is the convex hull of the set $\Delta_{Y_{n-k}\bullet}(D)$ and the point (x_1, \dots, x_k) . Note that we can fix a small neighborhood U of (x_1, \dots, x_k) in \mathbb{R}^k which is disjoint from $\Delta_{Y_{n-k}\bullet}(D + \varepsilon A)$.

There exists a sufficiently small $\delta > 0$ such that the divisors

$$\begin{aligned} A_1 &= A_1(\delta_1) \sim_{\mathbb{Q}} \frac{1}{2}\varepsilon A + \delta_1 Y_1, \\ A_2 &= A_2(\delta_1, \delta_2) \sim_{\mathbb{Q}} A_1|_{Y_1} + \delta_2 Y_2, \\ &\vdots \\ A_{n-k} &= A_{n-k}(\delta_1, \delta_2, \dots, \delta_{n-k}) \sim_{\mathbb{Q}} A_{n-k-1}|_{Y_{n-k-1}} + \delta_{n-k} Y_{n-k} \end{aligned}$$

are successively ample for any δ_j satisfying $\delta \geq \delta_1, \delta_2, \dots, \delta_{n-k} > 0$. Since $(0^{n-k}, x_1, \dots, x_k) \in \Delta_{Y_\bullet}(D)$, there exists a sequence of valuative points

$$\mathbf{x}_i = (\delta_1^i, \dots, \delta_{n-k}^i, x_1^i, \dots, x_k^i) \in \Delta_{Y_\bullet}(D)$$

such that

$$\lim_{i \rightarrow \infty} \delta_j^i = 0 \text{ for } 1 \leq j \leq n-k \text{ and } \lim_{i \rightarrow \infty} x_l^i = x_l \text{ for } 1 \leq l \leq k.$$

Since it is known that the set of rational valuative points $\{\nu_{Y_\bullet}(D') | D \sim_{\mathbb{Q}} D' \geq 0\}$ is dense in $\Delta_{Y_\bullet}(D)$, we may assume that $\mathbf{x}_i \in \{\nu_{Y_\bullet}(D') | D \sim_{\mathbb{Q}} D' \geq 0\}$ so that $\mathbf{x}_i \in \mathbb{Q}^n$ for all i . We now fix a sufficiently large i such that $0 \leq \delta_j^i < \delta$ for all $1 \leq j \leq n-k$ and (x_1^i, \dots, x_k^i) lies in the

small neighborhood U in \mathbb{R}^k of (x_1, \dots, x_k) . Since \mathbf{x}_i is a rational valutive point of $\Delta_{Y_\bullet}(D)$, there exist an effective divisor $D' \sim_{\mathbb{Q}} D$ such that $\nu_{Y_\bullet}(D') = \mathbf{x}_i$. Namely, we have

$$\begin{aligned} D' &= D_1 + \delta_1^i Y_1, \\ D_1|_{Y_1} &= D_2 + \delta_2^i Y_2, \\ &\vdots \\ D_{n-k-1}|_{Y_{n-k-1}} &= D_{n-k} + \delta_{n-k}^i Y_{n-k} \end{aligned}$$

where D_j on Y_{j-1} ($j = 1, \dots, n-k$) are effective divisors.

Now note that we have

$$D' + \frac{1}{2}\varepsilon A = D_1 + \left(\frac{1}{2}\varepsilon A + \delta_1^i Y_1\right) \sim_{\mathbb{Q}} D_1 + A'_1$$

where we may assume that A'_1 is an effective ample divisor such that $\text{mult}_{Y_1} A'_1 = 0$. We also have

$$(D_1 + A'_1)|_{Y_1} = D_2 + (A'_1|_{Y_1} + \delta_2^i Y_2) \sim_{\mathbb{Q}} D_2 + A'_2$$

where we may assume that A'_2 is an effective ample divisor such that $\text{mult}_{Y_2} A'_2 = 0$. By continuing this process, we finally obtain

$$(D_{n-k-1} + A'_{n-k-1})|_{Y_{n-k-1}} = D_{n-k} + (A'_{n-k-1}|_{Y_{n-k-1}} + \delta_{n-k}^i Y_{n-k}) \sim_{\mathbb{Q}} D_{n-k} + A'_{n-k}$$

where we may assume that A'_{n-k} is an effective ample divisor such that $\text{mult}_{Y_{n-k}} A'_{n-k} = 0$.

We now claim that there exists an effective divisor $D'' \sim_{\mathbb{Q}} D + \varepsilon A$ such that $D''|_{Y_{n-k-1}} = D_{n-k} + E$ for some effective divisor E with $\text{mult}_{Y_{n-k}} E = 0$ and $\nu_{Y_{n-k}\bullet}(E|_{Y_{n-k}}) = (x'_1, \dots, x'_k)$ where we may assume that $x'_j \geq 0$ are arbitrarily small. Note that such D'' defines a rational valutive point $\nu_{Y_\bullet}(D'') = (0^{n-k}, x'_1 + x'_1, \dots, x'_k + x'_k) \in \Delta_{Y_\bullet}(D + \varepsilon A)$. Thus $(x'_1 + x'_1, \dots, x'_k + x'_k) \in \Delta_{Y_{n-k}\bullet}(D + \varepsilon A)$. If our claim holds, then we can conclude that $(x'_1 + x'_1, \dots, x'_k + x'_k)$ belongs to the small neighborhood U of (x_1, \dots, x_k) in \mathbb{R}^k , which is a contradiction since U is disjoint from $\Delta_{Y_{n-k}\bullet}(D + \varepsilon A)$. Therefore we finally obtain $\Delta_{Y_{n-k}\bullet}(D) = \Delta_{Y_\bullet}(D) \cap (\{0\}^{n-k} \times \mathbb{R}_{\geq 0}^k)$.

It now remains to show the claim. For a sufficiently divisible and large integer $m > 0$, we take a log resolution $f_m: \tilde{X}_m \rightarrow X$ of the base ideal of $|m(D + \frac{1}{2}\varepsilon A)|$ so that we obtain a decomposition $f_m^*(m(D + \frac{1}{2}\varepsilon A)) = M'_m + F'_m$ into a base point free divisor M'_m and the fixed part F'_m of $|f_m^*(m(D + \frac{1}{2}\varepsilon A))|$. Let $M_m := \frac{1}{m}M'_m$. We may assume that f_m is isomorphic outside $\mathbf{B}_+(D + \frac{1}{2}\varepsilon A)$. We can take smooth strict transforms \tilde{Y}_i^m on \tilde{X}_m of Y_i for $1 \leq i \leq n-k$. For a general point y in \tilde{Y}_{n-k}^m , we have the positive moving Seshadri constant $\varepsilon(|D + \frac{1}{2}\varepsilon A|; f_m(y)) > 0$. Thus we also have the positive Seshadri constant $\varepsilon(M_m; y) > 0$ for $m \gg 0$ so that $\tilde{Y}_{n-k}^m \not\subset \mathbf{B}_+(M_m)$. Let $g_m: \tilde{X}_m \rightarrow Z_m$ be the birational morphism defined by $|M'_m|$. Possibly by taking a further blow-up of \tilde{X}_m , we may assume that every irreducible component of the exceptional locus of g_m is a divisor. We can still assume that f_m is isomorphic over a general point in Y_{n-k} . The divisor $H_m := M_m - E_m$ is ample for any sufficiently small effective divisor E_m whose support is the g_m -exceptional locus. Note that $\text{mult}_{\tilde{Y}_{n-k}^m}(E_m) = 0$. Let $f_m^*(D + \frac{1}{2}\varepsilon A) = P_m + N_m$ be the divisorial Zariski decomposition. As in [Le1, Proof of Proposition 3.7], by applying [ELMNP1, Proposition 2.5], we see that $P_m - M_m$ is arbitrarily small if we take a sufficiently large $m > 0$. Since we may take an arbitrarily small E_m , so is $P_m - H_m$ for a sufficiently large $m > 0$.

For simplicity, we fix a sufficiently large integer $m > 0$ and we denote $f = f_m$, $\tilde{X} = \tilde{X}_m$ and $\tilde{Y}_i = \tilde{Y}_i^m$. Let $f^*(D + \frac{1}{2}\varepsilon A) = P + N$ be the divisorial Zariski decomposition. Then as we have seen above, we can assume that P can be arbitrarily approximated by an ample divisor H on \tilde{X} such that $F = f^*(D + \frac{1}{2}\varepsilon A) - H$ is an effective divisor satisfying $\text{mult}_{\tilde{Y}_{n-k}}(F) = 0$. Note that $F - N$ is an arbitrarily small effective divisor such that $\text{mult}_{\tilde{Y}_{n-k}}(F - N) = 0$. Thus we can find an effective divisor $A_0 \sim_{\mathbb{Q}} A$ such that $\text{mult}_{Y_{n-k-1}} A_0 = 0$, $E_0 := \frac{1}{2}\varepsilon f^* A_0|_{\tilde{Y}_{n-k-1}} - (F - N)|_{\tilde{Y}_{n-k-1}}$ is

effective, and $\text{mult}_{\tilde{Y}_{n-k}} E_0 = 0$. Let $f^*D = P' + N'$ be the divisorial Zariski decomposition. Since $P' + f^*(\frac{1}{2}\varepsilon A)$ is movable, we get $P \geq P' + f^*(\frac{1}{2}\varepsilon A)$ and so $N' \geq N$. Since $Y_{n-k} \not\subseteq \mathbf{B}_+(D)$, every irreducible component of N' cannot contain \tilde{Y}_{n-k-1} . Clearly, $f^*D_{n-k} - N'|_{\tilde{Y}_{n-k-1}}$ is effective, and so is $f^*D_{n-k} - N|_{\tilde{Y}_{n-k-1}}$. Thus

$$E_1 := f^*(D_{n-k} + A'_{n-k}) - N|_{\tilde{Y}_{n-k-1}} + E_0 = f^*(D_{n-k} + A'_{n-k}) - F|_{\tilde{Y}_{n-k-1}} + \frac{1}{2}\varepsilon f^*A_0|_{\tilde{Y}_{n-k-1}}$$

is an effective divisor on \tilde{Y}_{n-k-1} . Note that $E_1 \sim_{\mathbb{Q}} (H + \frac{1}{2}\varepsilon f^*A)|_{\tilde{Y}_{n-k-1}}$. Since

$$H^0\left(\tilde{X}, m\left(H + \frac{1}{2}\varepsilon f^*A\right)\right) \rightarrow H^0\left(\tilde{Y}_{n-k-1}, m\left(H + \frac{1}{2}\varepsilon f^*A\right)\Big|_{\tilde{Y}_{n-k-1}}\right)$$

is surjective for all sufficiently divisible integers $m > 0$, it follows that there exists $H' \sim_{\mathbb{Q}} H + \frac{1}{2}\varepsilon f^*A$ such that $H'|_{\tilde{Y}_{n-k-1}} = E_1$. Then we have

$$(H' + F)|_{\tilde{Y}_{n-k-1}} = E_1 + F|_{\tilde{Y}_{n-k-1}} = f^*D_{n-k} + E'$$

where

$$E' := f^*A'_{n-k} + (F - N)|_{\tilde{Y}_{n-k-1}} + E_0 = f^*A'_{n-k} + \frac{1}{2}\varepsilon f^*A_0|_{\tilde{Y}_{n-k-1}}$$

is an effective divisor. Note that $\text{mult}_{\tilde{Y}_{n-k}} E' = 0$. We may also assume that each $x'_j \geq 0$ is arbitrarily small in $\nu_{\tilde{Y}_{n-k}\bullet}(E'|_{\tilde{Y}_{n-k}}) = (x'_1, \dots, x'_k)$. By letting $D'' := f_*(H' + F) \sim_{\mathbb{Q}} D + \varepsilon A$ and $E := f_*E'$, we obtain the divisors satisfying the required properties. This shows the claim, and hence, we complete the proof. \square

4. NAKAYAMA SUBVARIETIES AND POSITIVE VOLUME SUBVARIETIES

In [CHPW1], we introduced Nakayama subvarieties and positive volume subvarieties of divisors. We now further study those subvarieties, and prove Theorem B(=Theorem 4.8) in this section. We first recall the definitions of those subvarieties.

Definition 4.1 ([CHPW1, Definitions 2.7 and 2.13]). Let D be an \mathbb{R} -divisor on X .

(1) When D is effective, a *Nakayama subvariety* of D is an irreducible subvariety $U \subseteq X$ such that $\dim U = \kappa(D)$ and for every integer $m \geq 0$ the natural map

$$H^0(X, [mD]) \rightarrow H^0(U, [mD|_U])$$

is injective (or equivalently, $H^0(X, \mathcal{I}_U \otimes \mathcal{O}_X([mD])) = 0$ where \mathcal{I}_U is an ideal sheaf of U in X).

(2) When D is pseudoeffective, a *positive volume subvariety* of D is an irreducible subvariety $V \subseteq X$ such that $\dim V = \kappa_\nu(D)$ and $\text{vol}_{X|V}^+(D) > 0$.

Remark 4.2. In [CHPW1], we required an additional condition $V \not\subseteq \mathbf{B}_-(D)$ for the definition of positive volume subvariety. However, we can drop this condition by Lemma 4.3. Note that $V \not\subseteq \mathbf{B}_-(D)$ does not imply $\text{vol}_{X|V}^+(D) > 0$ (see [CHPW1, Example 2.14]).

Lemma 4.3. *Let D be a pseudoeffective \mathbb{R} -divisor on X . If V is a positive volume subvariety of D , then $V \not\subseteq \mathbf{B}_-(D)$.*

Proof. If $V \subseteq \mathbf{B}_-(D)$, then there is a sequence $\{A_i\}$ of ample divisors on X such that $\lim_{i \rightarrow \infty} A_i = 0$ and $V \subseteq \text{SB}(D + A_i)$. Then $\text{vol}_{X|V}(D + A_i) = 0$, so $\text{vol}_{X|V}^+(D) = 0$. Thus V is not a positive volume subvariety of D . \square

Remark 4.4. Even if V is a positive volume subvariety of D , it is possible that $V \subseteq \text{SB}(D)$. For instance, consider a ruled surface S carrying a nef divisor D such that $D \cdot C > 0$ for every irreducible curve $C \subseteq S$, but D is not ample (see e.g., [La, Example 1.5.2]). Since $\kappa(D) = -\infty$, we have $\text{SB}(D) = S$. Thus every positive volume subvariety of D is contained in $\text{SB}(D)$.

Remark 4.5. When $\kappa(D) = 0$ (resp. $\kappa_\nu(D) = 0$), every point not in $\text{Supp}(D)$ (resp. $\mathbf{B}_-(D)$) is a Nakayama (resp. positive volume) subvariety of D . When $\kappa(D) > 0$, any $\kappa(D)$ -dimensional general subvariety (e.g., intersection of general ample divisors) is a Nakayama subvariety of D ([CHPW1, Proposition 2.9]). Similarly, when $\kappa_\nu(D) > 0$, any $\kappa_\nu(D)$ -dimensional intersection of sufficiently ample divisors is a positive volume subvariety of D ([CHPW1, Proposition 2.17]). In particular, we can always construct an admissible flag Y_\bullet on X containing a Nakayama subvariety of D or a positive volume subvariety of D such that Y_n is a general point in X .

The importance of such special subvarieties associated to divisors is that one can read off interesting asymptotic properties of divisors from Okounkov bodies with respect to admissible flags containing those subvarieties. The following theorem is the main result of [CHPW1], which can be regarded as a generalization of [LM, Theorem A].

Theorem 4.6 ([CHPW1, Theorems A and B]). *We have the following:*

- (1) *Let D be an effective \mathbb{R} -divisor on X . Fix an admissible flag Y_\bullet containing a Nakayama subvariety U of D such that Y_n is a general point in X . Then $\Delta_{Y_\bullet}^{\text{val}}(D) \subseteq \{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$ so that one can regard $\Delta_{Y_\bullet}^{\text{val}}(D) \subseteq \mathbb{R}^{\kappa(D)}$. Furthermore, we have*

$$\dim \Delta_{Y_\bullet}^{\text{val}}(D) = \kappa(D) \text{ and } \text{vol}_{\mathbb{R}^{\kappa(D)}}(\Delta_{Y_\bullet}^{\text{val}}(D)) = \frac{1}{\kappa(D)!} \text{vol}_{X|U}(D).$$

- (2) *Let D be a pseudoeffective \mathbb{R} -divisor on X , and fix an admissible flag Y_\bullet containing a positive volume subvariety V of D . Then $\Delta_{Y_\bullet}^{\text{lim}}(D) \subseteq \{0\}^{n-\kappa_\nu(D)} \times \mathbb{R}^{\kappa_\nu(D)}$ so that one can regard $\Delta_{Y_\bullet}^{\text{lim}}(D) \subseteq \mathbb{R}^{\kappa_\nu(D)}$. Furthermore, we have*

$$\dim \Delta_{Y_\bullet}^{\text{lim}}(D) = \kappa_\nu(D) \text{ and } \text{vol}_{\mathbb{R}^{\kappa_\nu(D)}}(\Delta_{Y_\bullet}^{\text{lim}}(D)) = \frac{1}{\kappa_\nu(D)!} \text{vol}_{X|V}^+(D).$$

Remark 4.7. To extract asymptotic properties of divisors from $\Delta_{Y_\bullet}^{\text{val}}(D)$ as in Theorem 4.6 (1), we need to assume that Y_n is a general point in X . When considering $\Delta_{Y_\bullet}^{\text{val}}(D)$ (resp. $\Delta_{Y_\bullet}^{\text{lim}}(D)$), we say that Y_n is *general* if Y_n is not contained in $\text{SB}(D)$ (resp. $\mathbf{B}_-(D)$) (see [LM, Lemma 2.6] and [CHPW1, Subsection 3.2]).

As an application of Theorem 4.6, we now prove the following Theorem B.

Theorem 4.8. *Let D be an \mathbb{R} -divisor on X . Fix an admissible flag Y_\bullet such that Y_n is a general point in X . We have the following:*

- (1) *If D is effective, then Y_\bullet contains a Nakayama subvariety of D if and only if $\Delta_{Y_\bullet}^{\text{val}}(D) \subseteq \{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$.*
- (2) *If D is pseudoeffective, then Y_\bullet contains a positive volume subvariety of D if and only if $\Delta_{Y_\bullet}^{\text{lim}}(D) \subseteq \{0\}^{n-\kappa_\nu(D)} \times \mathbb{R}^{\kappa_\nu(D)}$ and $\dim \Delta_{Y_\bullet}^{\text{lim}}(D) = \kappa_\nu(D)$.*

Proof. The (\Rightarrow) direction of both (1) and (2) at once follows from Theorem 4.6. For the (\Leftarrow) direction of (1), note that $\text{ord}_{Y_{n-\kappa(D)}}(D') = 0$ for every effective divisor $D' \sim_{\mathbb{R}} D$ under the assumption that $\Delta_{Y_\bullet}^{\text{val}}(D) \subseteq \{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$. This means that $H^0(X, \mathcal{I}_{Y_{n-\kappa(D)}} \otimes \mathcal{O}_X([mD])) = 0$ for every integer $m \geq 0$. Thus $Y_{n-\kappa(D)}$ is a Nakayama subvariety of D .

For the (\Leftarrow) direction of (2), take an arbitrary ample divisor A on X . Since $\Delta_{Y_\bullet}(D + A) \supseteq \Delta_{Y_\bullet}^{\text{lim}}(D)$, it follows that

$$\Delta_{Y_\bullet}(D + A) \cap (\{0\}^{n-\kappa_\nu(D)} \times \mathbb{R}_{\geq 0}^{\kappa_\nu(D)}) \supseteq \Delta_{Y_\bullet}^{\text{lim}}(D).$$

Since Y_n is general, we have $Y_{n-\kappa_\nu(D)} \not\subseteq \mathbf{B}_-(D)$. Thus $Y_{n-\kappa_\nu(D)} \not\subseteq \mathbf{B}_+(D + A)$ and using Theorem 3.6, we obtain $\Delta_{Y_{n-\kappa_\nu(D)} \bullet}(D + A) \supseteq \Delta_{Y_\bullet}^{\text{lim}}(D)$. Therefore, by [LM, (2.7)] we have

$$\begin{aligned} \text{vol}_{X|Y_{n-\kappa_\nu(D)}}(D + A) &= \kappa_\nu(D)! \cdot \text{vol}_{\mathbb{R}^{\kappa_\nu(D)}} \Delta_{Y_{n-\kappa_\nu(D)} \bullet}(D + A) \\ &\geq \kappa_\nu(D)! \cdot \text{vol}_{\mathbb{R}^{\kappa_\nu(D)}} \Delta_{Y_\bullet}^{\text{lim}}(D). \end{aligned}$$

The given condition implies that $\text{vol}_{\mathbb{R}\kappa_\nu(D)} \Delta_{Y_\bullet}^{\lim}(D) > 0$. Hence, $\text{vol}_{X|Y_{n-\kappa_\nu(D)}}^+(D) > 0$, and by definition $Y_{n-\kappa_\nu(D)}$ is a positive volume subvariety of D . \square

Regarding Theorem 4.8 (1), we recall that $\dim \Delta_{Y_\bullet}^{\text{val}}(D) = \kappa(D)$ always holds whenever D is effective by [B2, Proposition 3.3].

5. RATIONAL POLYHEDRALITY OF OKOUNKOV BODIES

This section is devoted to showing the rational polyhedrality of Okounkov bodies of pseudo-effective divisors. We then finally prove Theorem C (=Corollary 5.4 and Theorem 5.6). First, we study the Okounkov bodies under surjective morphisms.

Lemma 5.1 (cf. [CHPW2, Lemma 3.3]). *Let $f: X \rightarrow \overline{X}$ be a surjective morphism of projective varieties of the same dimension n , and fix an admissible flag*

$$Y_\bullet: X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{x\}$$

on X such that

$$\overline{Y}_\bullet: \overline{X} = f(Y_0) \supseteq f(Y_1) \supseteq \cdots \supseteq f(Y_{n-1}) \supseteq f(Y_n) = \{f(x)\}$$

*is an admissible flag on \overline{X} . For a big \mathbb{Z} -divisor D on \overline{X} , consider a graded linear series W_\bullet associated to f^*D on X with $W_k := H^0(\overline{X}, kD) \subseteq H^0(X, kf^*D)$ for any integer $k \geq 0$. Then $\Delta_{Y_\bullet}(W_\bullet) = \Delta_{\overline{Y}_\bullet}(D)$.*

Proof. It follows from the construction of Okounkov body associated to a graded linear series. \square

The following lemma plays a crucial role in proving Theorem C.

Lemma 5.2 (cf. [AKL, Proposition 4]). *Let W_\bullet be a graded linear series on a smooth projective variety X generated by a base point free linear series W_1 . Suppose also that W_1 defines a surjective morphism $f: X \rightarrow \overline{X}$ of projective varieties of the same dimension n . Let Y_\bullet be an admissible flag on X defined by successive intersection of sufficiently general members E_1, \dots, E_n of W_1 ; $Y_i := E_1 \cap \cdots \cap E_i$ for $1 \leq i \leq n-1$ and $Y_n = \{x\}$ is a general point in X . Then $\Delta_{Y_\bullet}(W_\bullet)$ is a n -dimensional simplex in $\mathbb{R}_{\geq 0}^n$ whose vertices are $0, e_1, \dots, e_{n-1}, \text{vol}_X(W_\bullet)e_n$.*

Proof. There exists a very ample \mathbb{Z} -divisor D on \overline{X} so that we may assume $W_k = H^0(\overline{X}, kD) \subseteq H^0(X, kf^*D)$ for any integer $k \geq 0$. By the genericity assumption on E_j for defining Y_i , we may assume that

$$\overline{Y}_\bullet: \overline{X} = f(Y_0) \supseteq f(Y_1) \supseteq \cdots \supseteq f(Y_{n-1}) \supseteq f(Y_n)$$

is an admissible flag on \overline{X} . By Lemma 5.1, $\Delta_{Y_\bullet}(W_\bullet) = \Delta_{\overline{Y}_\bullet}(D)$. Note that $D^n = \text{vol}_{\overline{X}}(D) = \text{vol}_X(W_\bullet)$. By applying [AKL, Proposition 4] to $\Delta_{\overline{Y}_\bullet}(D)$, we obtain the assertion. \square

We now show the rational polyhedrality of $\Delta_{Y_\bullet}^{\text{val}}(D)$.

Theorem 5.3. *Let D be an effective \mathbb{Q} -divisor on X with finitely generated section ring $R(X, D)$. Then there exists an admissible flag Y_\bullet on X containing a Nakayama subvariety of D such that $\Delta_{Y_\bullet}^{\text{val}}(D)$ is a rational simplex in $\{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$ of dimension $\kappa(D)$.*

Proof. Let $m > 0$ be a sufficiently divisible and large integer such that mD is a \mathbb{Z} -divisor and the section ring $R(X, mD)$ is generated by $H^0(X, mD)$. We take a log resolution $f: \tilde{X} \rightarrow X$ of the base ideal $\mathfrak{b}(|mD|)$ so that we obtain a decomposition $f^*(mD) = M + F$ into a base point free divisor M and the fixed part F of $|f^*(mD)|$. Note that the morphism $\phi: \tilde{X} \rightarrow Z$ given by $|M|$ is the Iitaka fibration of f^*D . Let $A_1, \dots, A_{n-\kappa(D)}$ be sufficiently general ample divisors on \tilde{X} such that each $Y'_i := A_1 \cap \cdots \cap A_i$ for $1 \leq i \leq n-\kappa(D)$ is a smooth irreducible subvariety of dimension $n-i$. By Remark 4.5, $U := Y'_{n-\kappa(D)}$ is a Nakayama subvariety of f^*D . Let W_k be the image of the natural injective map $H^0(\tilde{X}, kf^*(mD)) \rightarrow H^0(U, kf^*(mD)|_U)$ for any integer $k \geq 0$. Then W_\bullet

is a graded linear series on U generated by W_1 . Note that $\phi|_U: U \rightarrow Z$ is a surjective morphism of projective varieties of the same dimension $\kappa(D)$ defined by W_1 . Now take sufficiently general members $E_1, \dots, E_{\kappa(D)}$ of W_1 such that $Y'_{n-\kappa(D)+i} := E_1 \cap \dots \cap E_i$ for $1 \leq i \leq \kappa(D) - 1$ is a smooth irreducible subvariety of X (and U) of dimension $\kappa(D) - i$, and $Y'_n = \{x\}$ where x is a general point in U . In particular, $Y'_\bullet: Y'_0 \supseteq \dots \supseteq Y'_n$ is an admissible flag on \tilde{X} and the partial flag $Y'_{n-\kappa(D)\bullet}$ is an admissible flag on U . Then by Lemma 5.2, $\Delta_{Y'_{n-\kappa(D)\bullet}}(W_\bullet)$ is a $\kappa(D)$ -dimensional simplex. Recall from [CHPW1, Remark 3.11] that $\Delta_{Y'_\bullet}^{\text{val}}(f^*D) = \Delta_{Y'_{n-\kappa(D)\bullet}}(W_\bullet)$. Furthermore, by the genericity assumption on Y'_\bullet , we can assume that $Y_\bullet: f(Y'_0) \supseteq \dots \supseteq f(Y'_n)$ is an admissible flag on X and $f(Y'_{n-\kappa(D)})$ is a Nakayama subvariety of D . By Lemma 3.4, $\Delta_{Y_\bullet}^{\text{val}}(D) = \Delta_{Y'_\bullet}^{\text{val}}(f^*D)$, and hence, $\Delta_{Y_\bullet}^{\text{val}}(D)$ is a rational simplex. Finally, by Theorem 4.6 (1), $\Delta_{Y_\bullet}^{\text{val}}(D)$ is contained in $\{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$ and is of dimension $\kappa(D)$. \square

Corollary 5.4. *Let D be an effective \mathbb{Q} -divisor on X which admits the birational good Zariski decomposition. Then there exists an admissible flag Y_\bullet on X containing a Nakayama subvariety of D such that $\Delta_{Y_\bullet}^{\text{val}}(D)$ is a rational simplex in $\{0\}^{n-\kappa(D)} \times \mathbb{R}^{\kappa(D)}$ of dimension $\kappa(D)$.*

Proof. By Proposition 2.7, D has a finitely generated section ring. Then the assertion now follows from Theorem 5.3. \square

We now turn to the limiting Okounkov body case.

Lemma 5.5. *Let P be a nef divisor on X , and consider an admissible flag Y_\bullet on X containing a smooth positive volume subvariety $V = Y_{n-\kappa_\nu(D)}$ of P . Then $\Delta_{Y_\bullet}^{\text{lim}}(P) = \Delta_{Y_{n-\kappa_\nu(D)\bullet}}(P|_V)$.*

Proof. By definition, it is clear that $\Delta_{Y_\bullet}^{\text{lim}}(P) \supseteq \Delta_{Y_{n-\kappa_\nu(D)\bullet}}(P|_V)$. Thus it is sufficient to show that their Euclidean volumes in $\mathbb{R}^{\kappa_\nu(P)}$ are equal, i.e., $\text{vol}_{\mathbb{R}^{\kappa_\nu(P)}}(\Delta_{Y_\bullet}^{\text{lim}}(P)) = \text{vol}_{\mathbb{R}^{\kappa_\nu(P)}}(\Delta_{Y_{n-\kappa_\nu(D)\bullet}}(P|_V))$, or equivalently, $\text{vol}_{X|V}^+(P) = \text{vol}_V(P|_V)$ by Theorem 4.6. Fix an ample divisor A on X . Since $P + \varepsilon A$ is ample for any $\varepsilon > 0$, it follows that $\text{vol}_{X|V}(P + \varepsilon A) = \text{vol}_V((P + \varepsilon A)|_V)$. By the continuity of the volume function, we obtain

$$\text{vol}_{X|V}^+(P) = \lim_{\varepsilon \rightarrow 0+} \text{vol}_{X|V}(P + \varepsilon A) = \lim_{\varepsilon \rightarrow 0+} \text{vol}_V((P + \varepsilon A)|_V) = \text{vol}_V(P|_V),$$

so we complete the proof. \square

We next obtain an analogous result on the rational polyhedrality of $\Delta_{Y_\bullet}^{\text{lim}}(D)$.

Theorem 5.6. *Let D be a pseudoeffective \mathbb{Q} -divisor on X which admits the birational good Zariski decomposition. Then there exists an admissible flag Y_\bullet on X containing a positive volume subvariety of D such that $\Delta_{Y_\bullet}^{\text{lim}}(D)$ is a rational simplex in $\{0\}^{n-\kappa_\nu(D)} \times \mathbb{R}^{\kappa_\nu(D)}$ of dimension $\kappa_\nu(D)$.*

Proof. Let $f: \tilde{X} \rightarrow X$ be a birational morphism of smooth projective varieties of dimension n such that $f^*D = P + N$ is the good Zariski decomposition. Let $A_1, \dots, A_{n-\kappa_\nu(D)}$ be sufficiently general ample divisors on \tilde{X} such that each $Y'_i := A_1 \cap \dots \cap A_i$ for $1 \leq i \leq n - \kappa_\nu(D)$ is a smooth irreducible subvariety of dimension $n - i$. By Remark 4.5, $V := Y'_{n-\kappa_\nu(D)}$ is a positive volume subvariety of f^*D . By [CHPW1, Theorem 2.18], $P|_V$ is big, and $mP|_V$ on V is base point free for a sufficiently divisible and large integer $m > 0$. Let $E_1, \dots, E_{\kappa_\nu(D)-1} \in |mP|_V|$ be general members such that each $Y'_{n-\kappa_\nu(D)+i} := E_1 \cap \dots \cap E_i$ for $1 \leq i \leq \kappa_\nu(D) - 1$ is a smooth irreducible subvariety of X of dimension $\kappa_\nu(D) - i$, and $Y'_n := \{x\}$ where x is a general point in V . Then $Y'_\bullet: \tilde{X} = Y'_0 \supseteq \dots \supseteq Y'_n$ is an admissible flag on \tilde{X} . By [AKL, Theorem 7], $\Delta_{Y'_{n-\kappa_\nu(D)\bullet}}(P|_V)$ is a $\kappa_\nu(D)$ -dimensional simplex. By Lemma 5.5, $\Delta_{Y_\bullet}^{\text{lim}}(P) = \Delta_{Y'_{n-\kappa_\nu(D)\bullet}}(P|_V)$, and by Lemma 3.5, $\Delta_{Y'_\bullet}^{\text{lim}}(f^*D) = \Delta_{Y'_\bullet}^{\text{lim}}(P)$. By the genericity assumption on Y'_\bullet , we can assume

that $Y_\bullet : f(Y'_0) \supseteq \cdots \supseteq f(Y'_n)$ is an admissible flag on X and $f(Y'_{n-\kappa_\nu(D)})$ is a positive volume subvariety of D . By Lemma 3.4, we obtain $\Delta_{Y_\bullet}^{\lim}(D) = \Delta_{Y'_\bullet}^{\lim}(f^*D)$, and hence, $\Delta_{Y_\bullet}^{\lim}(D)$ is a rational simplex. Finally, by Theorem 4.6, $\Delta_{Y_\bullet}^{\lim}(D)$ is in $\{0\}^{n-\kappa_\nu(D)} \times \mathbb{R}^{\kappa_\nu(D)}$ and of dimension $\kappa_\nu(D)$. \square

Remark 5.7. The problem of the rational polyhedrality of Okounkov body is not yet fully understood. It was shown in [AKL, Corollary 13] and [CPW, Theorems 1.1 and 4.17] that on a smooth projective surface, there always exists an admissible flag with respect to which the Okounkov body of any \mathbb{Q} -divisor is a rational polytope. Thus, in particular, even if a pseudoeffective \mathbb{Q} -divisor is not abundant or does not have finitely generated section ring, the associated Okounkov body can still be a rational polytope with respect to some admissible flag. On the other hand, even when the given variety is a Mori dream space, the Okounkov body can be non-polyhedral for some admissible flag (see [KLM, Section 3]).

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